

Synchronization and directed percolation in coupled map lattices

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We study a synchronization mechanism, based on one-way coupling of all-or-nothing type, applied to coupled map lattices with several different local rules. By analyzing the metric and the topological distance between the two systems, we found two different regimes: a strong chaos phase in which the transition has a directed percolation character and a weak chaos phase in which the synchronization transition occurs abruptly. We are able to derive some analytical approximations for the location of the transition point and the critical properties of the system. We propose to use the characteristics of this transition as indicators of the spatial propagation of chaoticity. [S1063-651X(99)06701-X]

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I. INTRODUCTION

Recently, the synchronization of chaotic systems has received considerable attention [1–7]. Among the many papers on this subject, some of them concern the behavior of spatially extended chaotic systems [8–10].

In this paper we study the synchronization properties of two coupled map lattices [11], when the coupling between them is completely asymmetric, and the synchronization mechanism is of the all-or-none kind: denoting one system as the master and the other as the slave, either an individual map of the slave system is completely synchronized to the corresponding map of the master system or it is left free. We study the annealed version of this kind of coupling by choosing at random a fixed fraction of sites to be synchronized at each time step, which constitutes the control parameter of the synchronization transition.

An alternative point of view originates from the problem of the characterization of chaos in high dimensional dynamical systems. We assume a rather unusual point of view: instead of considering the rate of divergence of the distance between the trajectory of a reference system (the master) and a perturbed one (the slave), a procedure that leads to the definition of the maximum Lyapunov exponent, we measure the “efforts” needed to make the slave system coalesce with the master.

Let us illustrate in detail this approach. The standard technique for studying the chaotic properties of such a system is that of measuring the response of the system to a perturbation. The conceptual experiment is the following: (i) at a certain time make a copy of the master system and slightly vary its state; (ii) let the system evolve for a small interval of time and (iii) measure the distance between the master and

the slave; (iv) renormalize the distance so that it is always small. The logarithmic average of the growth rate of the distance gives the maximum Lyapunov exponent. For a small distance, the leading contribution in the diverging rate comes from the maximum eigenvalue of the product of the Jacobian of the evolution function computed along the trajectory. By considering more than just one vector in the tangent space, and keeping them orthogonal, one can obtain the whole spectrum of Lyapunov exponents [12].

The Lyapunov spectrum, however, does not describe accurately the process of spatial propagation of chaos. We shall illustrate this point by considering diffusively coupled logistic map lattices at Ulam point. By analyzing the behavior of the Lyapunov spectrum, one sees that for a small coupling the spectrum is almost constant and positive. When increasing the coupling between maps, both the positive part of the Lyapunov spectrum and the maximum exponent decrease.

From a different point of view, the chaoticity (and the Lyapunov exponent) of a single map can be defined by means of the “efforts” needed to synchronize the slave with the master. Let us consider the following “toy” system composed of two simple maps:

$$x' = f(x), \tag{1}$$

$$y' = (1-p)f(y) + pf(x),$$

where $x \equiv x(t)$ is the master system and $y \equiv y(t)$ is the slave [14]; the prime denotes the value of the map at time $t+1$. The master system x evolves freely while the slave y is subjected to two opposed contributions: it tends to separate from the master if the map f is chaotic, but it is pushed towards x by the parameter p , which represents the “strength.”

For a small difference $z(t) = x(t) - y(t)$, one has

$$z' = (1-p)f'(x)z, \tag{2}$$

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i.e., the synchronization $[z(\infty)=0]$ occurs for $p \geq p_c = 1 - \exp(-\lambda)$, where λ is the Lyapunov exponent of the (unperturbed) trajectory $x(t)$ of the master system. Thus for simple maps the synchronization threshold is related to the chaoticity of the system. We would like to extend this concept to a spatially extended system with a diffusive character.

The basic idea of our approach is to consider Eq. (1) as a mean-field description of a stochastic process which, with probability p , sets each individual map of the slave system to the value of the corresponding map of the master one (“pinching”). We assume that the synchronized state is absorbing, i.e., once a patch of the slave system is in the same state of the master one, desynchronization can occur only at borders, without bubbling. The completely synchronized state, even if unstable, cannot be exited. A discussion about the robustness of this synchronization mechanism is deferred to the final section.

This method can be applied also to more exotic dynamical systems, such as cellular automata [13]. For such systems the usual chaotic indicators cannot be easily computed, but they exhibit spatial propagation of disorder.

It is clear that for $p=1$ the system synchronizes in just one time step. However, as we shall show in detail in Sec. III, a critical value $p=p_c < 1$ does exist, above which the system synchronizes regardless of its chaoticity. This threshold is related to a directed percolation phase transition.

On the other hand, some systems can synchronize for $p=p^* < p_c$, because the propagation rate of the differences between the master and the slave is reduced by the inactivation of the degrees of freedom due to pinching. As a trivial illustration of this process, a lattice of *uncoupled* maps synchronizes in the long time limit for all strengths $p > p^* = 0$. This synchronization threshold is an indicator of the spatial propagation of chaoticity in the system. Other information come from the dynamics of the transition.

The sketch of this paper is as follows. In the next section we describe precisely the model we use and introduce the observables, in Sec. III the synchronization transition is analyzed, and the connections with the directed percolation problem are discussed. In Sec. IV we present the phase diagrams for the synchronization transition for several well-known maps, and finally in the last section we discuss our main conclusions, including possible extensions and applications of the synchronization mechanism, and the relationships with other models found in the literature.

II. THE STOCHASTIC SYNCHRONIZATION MECHANISM

The state of the master system at a given time t is denoted as $X^t \in [0,1]^N$, and a component of X (a single map) is indicated as $X|_i \equiv x_i, i = 1, \dots, N$. The dynamics is defined as

$$\begin{aligned} x_i^{t+1} &= g_\epsilon(x_{i-1}^t, x_i^t, x_{i+1}^t) \\ &\equiv (1-2\epsilon)f(x_i^t) + \epsilon[f(x_{i-1}^t) + f(x_{i+1}^t)], \end{aligned} \quad (3)$$

with periodic boundary conditions and $0 \leq \epsilon \leq 1/2$. It can be considered as the discretization (in time and space) of a reaction-diffusion system, the ϵ coupling coming from the Laplacian operator. The dynamics of the slave system Y is given by

$$\begin{aligned} y_i^{t+1} &= [1-r_i^t(p)]g_\epsilon(y_{i-1}^t, y_i^t, y_{i+1}^t) \\ &\quad + r_i^t(p)g_\epsilon(x_{i-1}^t, x_i^t, x_{i+1}^t), \end{aligned} \quad (4)$$

where $r_i^t(p)$ is a random variable which assumes the value one with probability p , and zero otherwise. In other words, at each time step a fraction p of maps in the slave system is set to the same value of the corresponding map in the master system.

In vectorial notation one can write for the master system

$$X^{t+1} = G_\epsilon(X^t) = (I + \epsilon\Delta)F(X^t),$$

where Δ is the discrete Laplacian

$$\Delta X|_{ij} = (\delta_{ij-1} + \delta_{ij+1} - 2\delta_{ij})x_j$$

and $F(X)$ is a diagonal operator $F(X)|_{ij} = f(x_j)\delta_{ij}$. For the slave system one has

$$Y^{t+1} = \bar{S}^t(p)G_\epsilon(Y^t) + S^t(p)G_\epsilon(X^t),$$

where $S(p)$ is a random diagonal matrix having a fraction p of diagonal elements equal to one and all others equal to zero, $S(p)^t|_{ij} = r_j^t(p)\delta_{ij}$, and $\bar{S}(p) = I - S(p)$.

We introduce also the difference system $Z = X - Y$, whose evolution rule is

$$Z^{t+1} = \bar{S}^t(p)[G_\epsilon(X^t) - G_\epsilon(X^t - Z^t)].$$

If the difference field is uniformly small, i.e., for $\max(|z_i^t|) \rightarrow 0$, one has

$$Z^{t+1} = \bar{S}^t(p)J_\epsilon(X^t)Z^t,$$

where $J_\epsilon(X)$ is the Jacobian of the evolution function

$$J_\epsilon(X^t) = (I + \epsilon\Delta) \frac{\partial F(X^t)}{\partial X^t}$$

and

$$\left. \frac{\partial F(X)}{\partial X} \right|_{ij} = \delta_{ij} \left. \frac{df(x)}{dx} \right|_{x=x_j}.$$

For all kinds of numerical computations there is a limit to the precision below which two numbers become indistinguishable. Since this limit depends on the magnitude of the numbers, it is very hard to control its effects. In this perspective, we introduce a threshold τ on the precision, by imposing that if $|x_i - y_i| < \tau$, then $y_i = x_i$. In this way we can study the sensitivity of the results on τ , and eventually perform the limit $\tau \rightarrow 0$. We have checked that our asymptotic results are independent of τ , at least for τ smaller than 10^{-6} . In the following, we shall neglect indicating the truncation operation for ease of notation, except when explicitly needed. Due to the precision threshold τ , the difference field z_i^t is set to zero if $z_i^t < \tau$.

An alternative way of computing the evolution of the system, which will be useful in the following, is given by the following procedure.

(i) Consider three stacked two-dimensional lattices of size $N \times T$, and label one direction as space i , $i \in [1, N]$ and the other one as time t , $t \in [0, T]$. This ensemble can be considered to be composed of three layers: one that contains the Boolean numbers r_i^t and two containing the real numbers x_i^t and y_i^t .

(ii) Fill up the r_i^t layer with zeros and ones so that the probability of having $r_i^t = 1$ is p ; this layer will be named the quenched field.

(iii) Fill up the x_i^0 and y_i^0 rows at random; they will be the initial conditions for the X and Y lattice maps.

(iv) Iterate the applications (3) and (4) to fill up the X and Y layers.

In this way we define an out-of-equilibrium statistical system, with p as a control parameter. It is assumed the limit $N \rightarrow \infty$ and $T \rightarrow \infty$ and the average over the quenched field. The degree of synchronization of the system at a given time t can be measured by the (metric) distance

$$\zeta^t = \frac{1}{N} \sum_{i=1}^N |z_i^t|,$$

whose asymptotic value for a given probability p will be denoted as $\zeta(p)$.

We introduce also the field h_i^t as

$$h_i^t = \begin{cases} 0 & \text{if } z_i^t = 0 \\ 1 & \text{otherwise,} \end{cases}$$

and the topological distance

$$\rho^t = \frac{1}{N} \sum_{i=1}^N h_i^t$$

which measures the fraction of nonsynchronized sites in the system. The asymptotic value of the topological distance will be denoted as $\rho(p)$.

We shall study the synchronization transition for the following maps $f(x)$ of the unit interval $x \in [0, 1]$.

(i) The generalized Bernoulli shift $f(x) = [ax] \bmod 1$ with a slope a greater than 1. The Lyapunov exponent of the single map is simply $\ln(a)$; this is also the value of the maximum Lyapunov exponent for a diffusively coupled lattice, regardless of ϵ .

(ii) The quenched random map that assumes a different (random) value for each different x . It can be considered equivalent to the Bernoulli shift in the limit $a \rightarrow \infty$, and thus the map with the highest degree of chaoticity. This map is everywhere nondifferentiable.

(iii) The logistic map $f(x) = ax(1-x)$, which is chaotic for $3.57 \dots < a \leq 4$.

(iv) The generalized tent map

$$f(x) = \left[a \left(\frac{1}{2} - \left| x - \frac{1}{2} \right| \right) \right] \bmod 1$$

with $a > 1$.

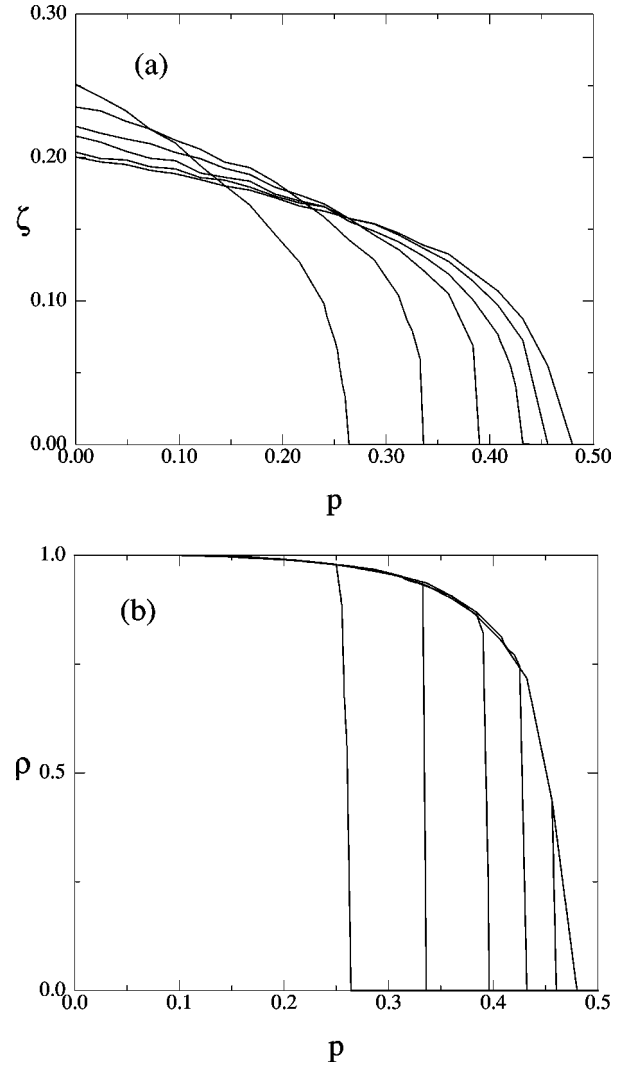


FIG. 1. Metric distance (a) and topological distance (b) for a chain of 32 000 Bernoulli maps $\epsilon = 1/3$ and $a = 1.2, 1.4, \dots, 2.2$ from left to right. One run of 30 000 time steps.

III. THE SYNCHRONIZATION TRANSITION

If the maps are uncoupled ($\epsilon = 0$), any value of p greater than zero is sufficient to synchronize the system in the long time limit, regardless of the chaoticity of the single map. For coupled systems, however, the chaoticity of the map (temporal chaos) contributes to the spatial chaoticity of the system.

For illustration purposes, we show in Fig. 1 the behavior of the metric distance $\zeta(p)$ (left) and the topological distance $\rho(p)$ (right) for Bernoulli maps with $\epsilon = 1/3$ and different values of a . The topological distance exhibits a sharp transition for $a < a_c \approx 2$ and a smooth transition for $a \geq a_c$; all curves superimpose to a universal curve far from the transition point. The metric distance always exhibits a smooth transition. This scenario is generic for all kinds of maps and couplings, except that some maps (notably the logistic map) never exhibit the smooth transition of the topological distance in the allowed range of values of the parameter a .

For the random map it is quite easy to understand the origin of the universal curve: since even a small distance is amplified in one time step to a random value, the difference z_i^t is greater than 0 if $z_j^{t-1} > 0$ on some of the neighbors

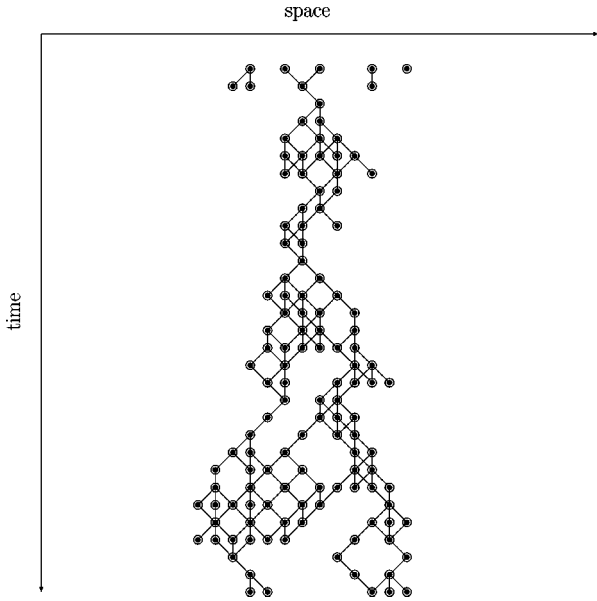


FIG. 2. An example of a wet cluster for $p=0.45$.

($|i-j|<1$) at the previous time step, and the quenched field r_i^t is equal to 0.

This defines a directed site percolation problem [15] (formation of a percolating cluster of “wet” sites along the time direction), with control parameter $1-p$. A site of coordinate (i,t) is said to be wet if $r_i^t(p)=0$ and it is connected at least to one neighboring wet site $r_j^{t-1}(p)=0$ ($|i-j|\leq 1$) at time $t-1$. All sites in the first ($t=0$) row are supposed to be connected to an external wet site. We shall denote the ensemble of wet sites by the term “wet cluster”; an instance of a wet cluster for $p=0.45$ is shown in Fig. 2. Notice that the usual directed percolation probability $p^{(\text{DP})}$ is given here by $p^{(\text{DP})}=1-p$.

If $p^{(\text{DP})}<p_c^{(\text{DP})}$, the wet cluster does not percolate in the time direction, and the system will synchronize despite any chaoticity of the single map. The $r_i^t=0$ sites disconnected from the percolation cluster do not influence the synchronization, since the synchronized state is locally absorbing. Thus, for the random map and $\epsilon>0$, the invariant curve is simply the curve of the asymptotic density $m(p)$ of wet sites for the directed percolation problem. In the vicinity of the transition, $m(p)$ behaves as

$$m(p)\sim(p_c-p)^\beta,$$

where $\beta\approx 0.26(1)$ is the magnetic critical exponents of the $(1+1)$ -dimensional directed percolation problem and $p_c=1-p_c^{(\text{DP})}\approx 0.460(2)$ for this lattice with connectivity three [16].

In the generic case, one has to study what happens on the wet cluster. If the evolution of the system is expanding in the difference space, and the wet cluster percolates in the time direction, then the asymptotic topological distance is greater than zero and vanishes when the density of wet sites does. The expansion rate in the difference space depends on the average number of connections between wet sites in the cluster, an exception being the random map, which expands to a random value regardless of the initial difference.

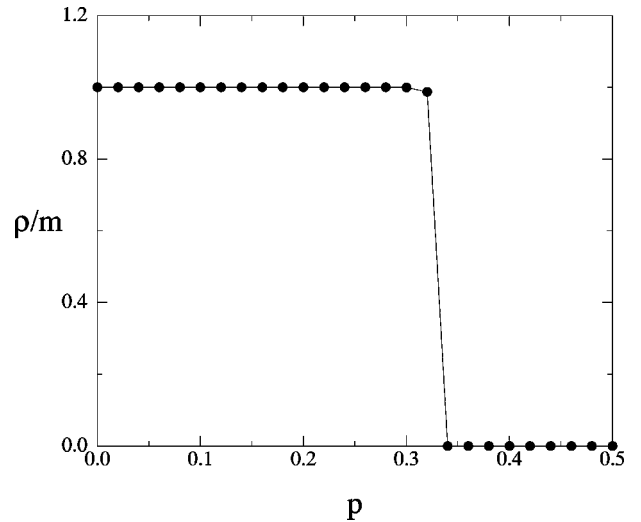


FIG. 3. The ratio $\rho(p)/m(p)$ vs p for a chain of 1000 logistic maps, $T=3000$, slope $a=4$, and $\epsilon=1/3$.

On the other hand, it may happen that for some value of p (for a given percolating wet cluster) the dynamics in the difference space is contracting, so that the difference field z_i^t will eventually become less than τ and thus set to zero. As an illustration, let us consider the Bernoulli shift for $\epsilon=1/3$ and only one connection on average. The problem reduces to the computation of the expansion rate in a one-dimensional chain of asymmetrically coupled maps, and the threshold for the synchronization is $a=3$.

This eventual contraction mechanism implies that the cluster of the sites for which $h_i^t=1$ (the difference cluster) is generally strictly included in the percolation cluster.

We classify the case in which the difference cluster always survives when the percolation cluster spans the lattice with the name *strong spatiotemporal chaos*, otherwise we have the *weak spatiotemporal chaos*. We shall denote with a star the values of the parameters for which the synchronization transition occurs, distinguishing the weak and strong cases by the index w or s , respectively.

From the results of our numerical simulations, we think that the following ansatz for the topological distance $\rho(p)$ holds: the strong chaos transition always belongs to the directed percolation universality class, while the weak chaos transition is always of first-order character. This statement is equivalent to the assumption that the fraction of sites in the percolation cluster that do not belong to the difference cluster is either vanishing (strong chaos) or order 1 (weak chaos) in the thermodynamic limit. If noise is allowed to desynchronize the system, the directed percolation character is lost.

This ansatz can be illustrated by plotting the ratio between the topological distance $\rho(p)$ and the density of wet sites $m(p)$ as shown in Fig. 3 for the logistic map and $\epsilon=1/3$. One can see that this ratio maintains almost constant up to the transition point. We have checked that this behavior holds for several values of the slope a and the coupling ϵ for the three maps.

It is interesting to examine the behavior of the difference z_i^t near the transition point. In the strong chaos phase, the value of the difference z_i^t for nonsynchronized sites is large, since the synchronization is forced by the percolation mecha-

nism. On the other hand, in the weak chaos phase, the value of the difference z_i^t for nonsynchronized sites is small on average, even though localized bubbling of large amplitude can be observed [17].

In the following we study numerically the phase boundary between the strong and weak chaos for the Bernoulli shift and the logistic and tent map, and we obtain some analytical approximation for it.

IV. PHASE DIAGRAMS

The boundary between the weakly and strongly chaotic phases can be defined considering the synchronization mechanism on the critical percolation cluster, i.e., the cluster of wet sites at $p^* = \min p_s^* = \max p_w^* = 1 - p_c^{(DP)}$. As illustrated in Appendix A, the critical wet cluster is included in all percolating wet clusters, so it can be considered to be the smallest percolating cluster of wet sites, and thus the one most unfavorable to the propagation of the difference, still presenting spanning paths over which chaos can propagate. The algorithm for the generation of the critical cluster is also described in Appendix A.

The phase diagrams for the Bernoulli shift and the tent map are shown in Fig. 4; the logistic map never exhibits the strongly chaotic phase. The case $\epsilon = 1/2$ is special, since the connectivity changes, and in this case the lattice corresponds to that of the Domany-Kinzel model [18], for which $1 - p_c^{(DP)} = 0.3$.

Since this boundary phase is defined on a critical percolation cluster, one can assume that the average expansion rate, given by the sum of all paths in the wet cluster, is dominated by a single path. Let us assume that this path contains x vertical steps (of weight $1 - 2\epsilon$) and y oblique steps (of weight ϵ). For the Bernoulli shift, all steps carry an expansion rate a and thus one obtains for the Lyapunov multiplier μ_c at the phase boundary

$$\mu_c \simeq a^{x+y} (1 - 2\epsilon)^x \epsilon^y \simeq 1$$

and thus

$$a^* = \frac{\text{const}}{(1 - 2\epsilon)^{\alpha_1} \epsilon^{\alpha_2}},$$

with

$$\alpha_1 + \alpha_2 = 1. \quad (5)$$

Indeed, the numerical data presented in Fig. 4 do support a correspondence of this form. However, they are too rough to allow the precise computation of the exponents, which at any rate do not seem to always correspond to Eq. (5).

We computed numerically the transition point p^* and the critical exponents β and ν_\perp for the topological distance $\rho(p)$ for the Bernoulli shift and the tent map by means of the scaling relation

$$\rho\left(p, \frac{1}{t}\right) = \alpha^{\beta/\nu_\perp} \cdot \rho\left(\alpha^{1/\nu_\perp}(p - p^*) + p^*, \frac{1}{\alpha t}\right) \quad (6)$$

that holds in the thermodynamic limit $N \rightarrow \infty$. Here α is an arbitrary time scaling factor. We found $p_c = 0.460(2)$, β

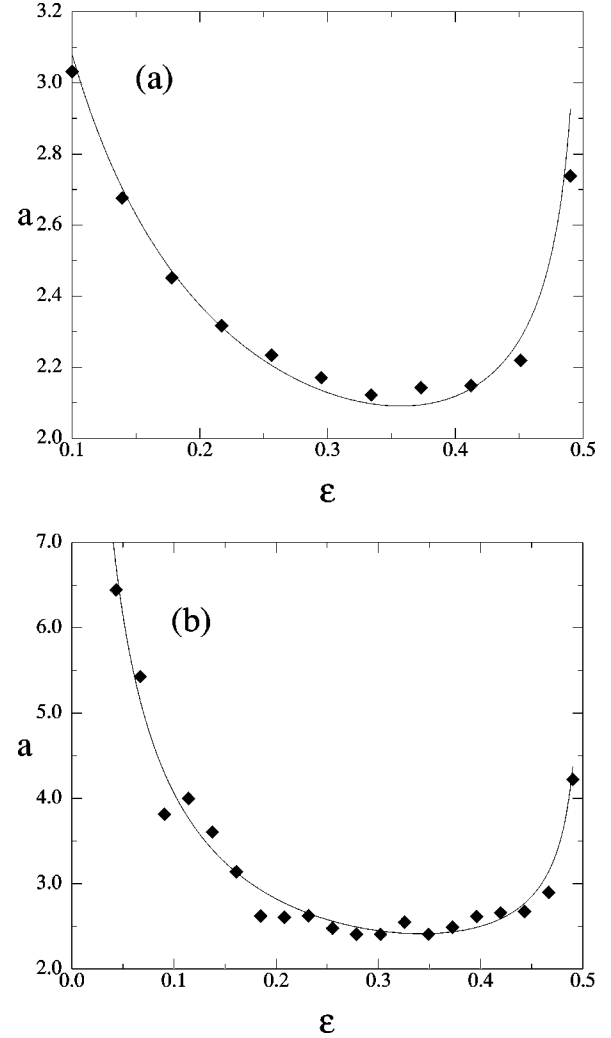


FIG. 4. Percolative phase diagram for Bernoulli (a) and tent (b) maps. The line represents a power-law fitting, as described in the text. Average over five runs, $N = 100$, $T = 4000$.

$= 0.26(1)$, and $\nu_\perp = 1.75(5)$. The results are consistent with the hypothesis that the phenomenon belongs to the directed percolation universality class, as expected.

The strong chaos transition is characterized by the expanding dynamics on the percolation cluster. Thus, in general, at the transition point the value of a nonzero difference z_i^t is not small. On the contrary, the weak chaos transition is characterized by the average vanishing of z_i^t . Up to the boundary between the two regimes, the distance field z_i is still small and a linear approximation can be used. The synchronization mechanism is related to the average number of links between wet sites, $\kappa(p)$. Since the wetting is forced (once a site has been wet, it cannot be ‘‘unwet’’), one can simply consider the process for a single site, obtaining $\kappa(p) = 3(1 - p)$.

By imposing that the distance should be marginally expanding, and assuming that the difference field z_i is almost constant, we get for the Bernoulli shift and $\epsilon = 1/3$

$$\frac{1}{3} a_w^* \kappa(p_w^*) = a_w^* (1 - p_w^*) = 1$$

and thus $a_w^* = 1/(1-p_w^*)$.

For a generic map, with positive and negative slopes, cancellation effects can be present. We need an indicator about the constancy of the sign of the derivative over the interaction range of the diffusion operator. We consider the quantity

$$\eta_i = \frac{f'(x_i) + f'(x_{i+1})}{2},$$

which is analogous to the local Lyapunov multiplier, and

$$\eta = \left(\prod_{i=0}^{N-1} \eta_i \right)^{1/N}$$

as a global indicator. For the Bernoulli map this indicator corresponds to the slope a .

Thus, for $\epsilon = 1/3$, the condition for observing the weak chaos synchronization transition is given in this approximation by

$$\frac{1}{3} \eta_w^* \kappa(p_w^*) = \eta_w^* (1-p_w^*) = 1.$$

We report in Fig. 5 the behavior of $\eta_w^* (1-p_w^*)$ as a function of the slope a for Bernoulli shift and tent maps, together with the mean-field approximation. The discrepancies from this approximation originate from the assumption of a uniformly vanishing difference field z_i , which is not fulfilled even at the transition point.

Let us turn now to the metric distance ζ . For random maps even an infinitesimally small distance is amplified in one step to a random value. Thus, we have for the metric distance

$$\zeta(t) = \langle |z| \rangle \rho(t),$$

and ζ has the same critical behavior of ρ , denoting a certain degree of universality in this synchronization transition.

We have studied the behavior of the ratio $\zeta(p)/\rho(p)$ as a function of p for various values of the slope a for the Bernoulli shift and the tent map, and $\epsilon = 1/3$. The results, reported in Fig. 6, show that for the Bernoulli shift the metric distance is independent of p and of slope a far from the synchronization transition. Deviations from this behavior near the transition point vanish as the slope a become greater than a^* .

For the tent map the ratio $\zeta(p)/\rho(p)$ is independent of p far for the transition but it depends on the slope a which determines the distribution of z_i . Again, for $a \gg a^*$, the metric distance is almost constant from p also near the transition point.

The case $\epsilon \neq 1/3$ is more difficult to analyze, since the asymmetric couplings cannot be easily mapped onto a statistical problem. However, for very small ϵ , the expansion is dominated by the exponential growth (with average rate λ) along the vertical link of the largest difference. A very rough mean-field description could be the following. Let us assume that at a certain time there is essentially only one site i with a non-null difference z_i . This difference grows exponentially at rate λ and propagates to the neighboring sites at rate ϵ . After an average time $1/p$ the difference at site i is set to zero by the synchronization mechanism, and after time $2/p$ only

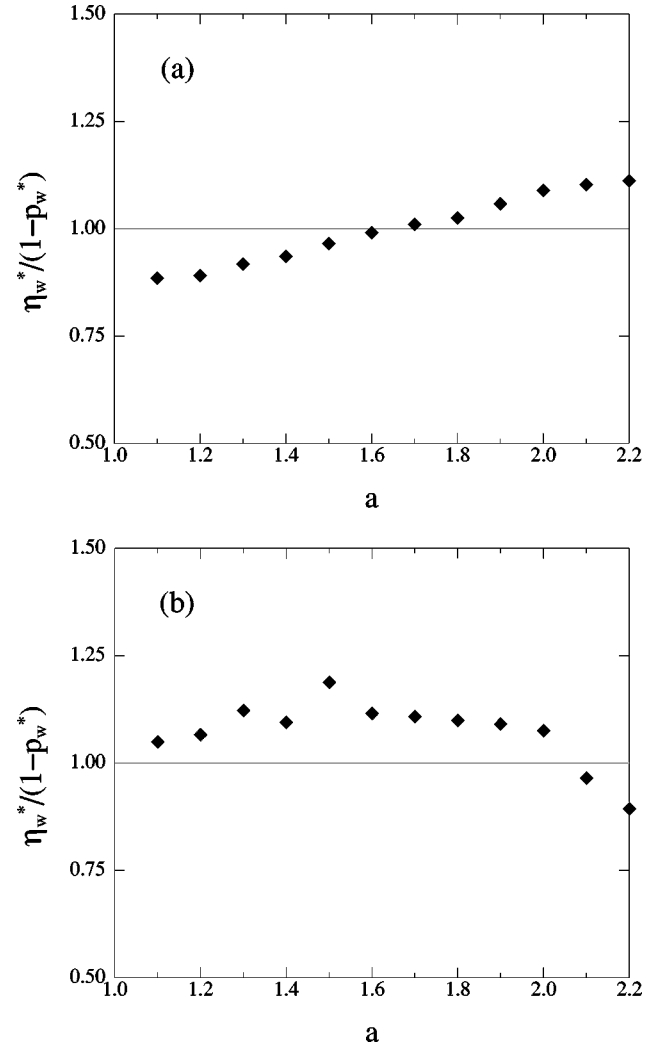


FIG. 5. The relation between the slope a and the weak chaos synchronization threshold $\eta_w^*/(1-p_w^*)$ for a chain of Bernoulli shift (a) and tent maps (b). Average over four runs, $N=400$, $T=8000$, and $\epsilon=1/3$.

one of the neighboring sites has a nonzero difference, thus closing the cycle. Therefore, the average expansion rate at the synchronization threshold p_c is

$$\epsilon \exp(2\lambda/p_c) \approx \text{const}, \quad (7)$$

where λ is the Lyapunov exponent of the uncoupled map, and we have neglected nonexponential prefactors. In Fig. 7 we show the results of one simulation for the three maps studied in the paper, with the parameters chosen so as to have $\lambda = \ln(2)$, and for Bernoulli maps with various slopes. One can see that Eq. (7) is verified for small ϵ , except finite size and time effects.

V. CONCLUSIONS

We have studied the synchronization transition between two chains of diffusively coupled chaotic maps, induced by the inactivation of degrees of freedom in the difference space with a probability p . We have found that two different regimes can be defined: the strong chaos regime for which the dynamics of the transition is dominated by the directed per-

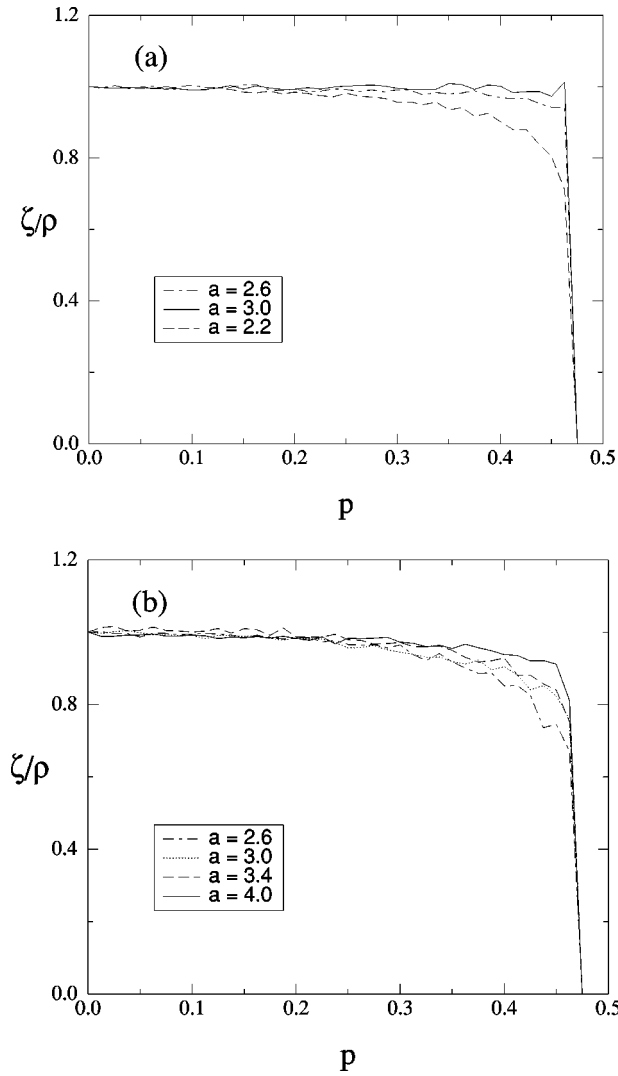


FIG. 6. The dependence of the ratio between the metric distance $\zeta(p)$ and the topological distance $\rho(p)$ as a function of p , for the Bernoulli shift (a) and tent map (b) and different slopes a . Average over four runs, $N=100$, $T=1000$. All curves normalized to 1 for $p=0$.

colation transition, and a weak chaos regime in which the system synchronizes in the presence of spanning paths along which the difference could in principle survive. We have been able to present some analytical approximation of the transition point and its critical properties.

The character of the transition and the critical value of the parameter p are proposed as indicators of spatial propagation of chaoticity, which can complement the usual Lyapunov description. These indicators do not rely on the existence of a tangent space or exponential growth, so they can be applied to a broader class of systems, such as nondifferentiable maps or cellular automata [13], and a system presenting stable chaos [19].

Our approach can be considered as the annealed version of models that exhibit a synchronization transition, presented in some recent papers. First of all, let us consider the synchronization mechanism proposed by Pecora and Carroll [1]. In their numerical and experimental setup they studied the behavior of the distance between two chaotic oscillators, when part of the degrees of freedom of one of them is set

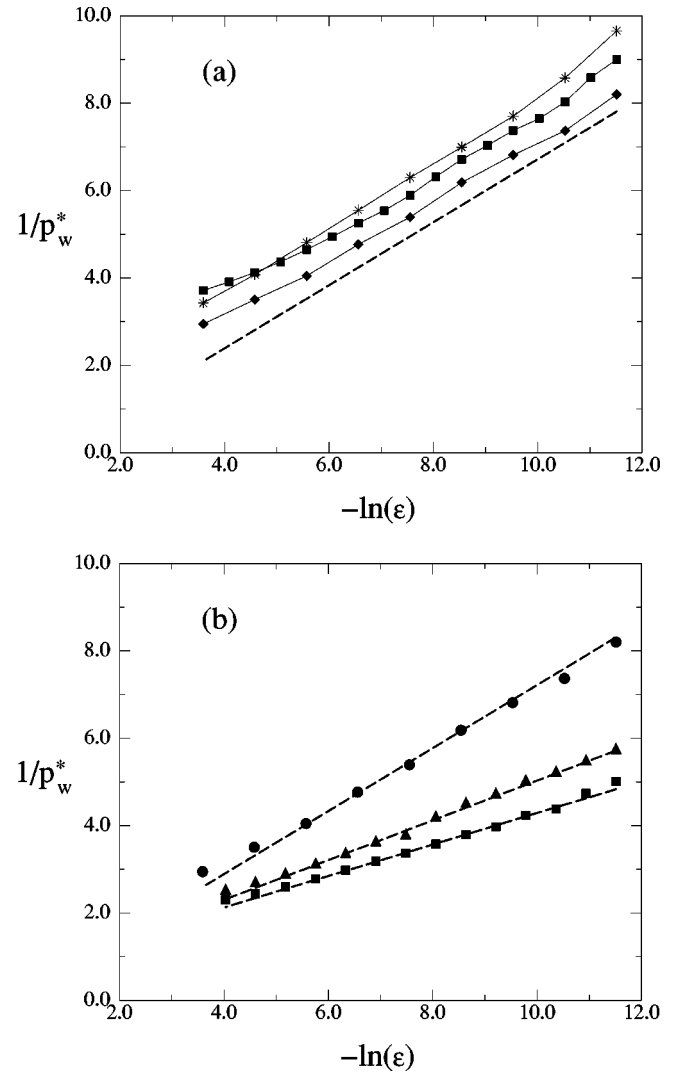


FIG. 7. Relation between a small coupling ϵ and the synchronization threshold p_w^* . (a) The three sets correspond to the Bernoulli shift with slope $a=2$ (stars), tent map with slope $a=2$ (diamonds), and logistic maps with $a=4$ (squares). For all three maps the maximum Lyapunov exponent for $\epsilon=0$ is $\lambda=\ln(2)$; the dashed line corresponds to the law $1/p_w^* = \ln(\epsilon)/[2\ln(2)]$. (b) Bernoulli shift for $a=2$ (circle), $a=3$ (triangles), and $a=4$ (squares); the dashed lines have slope $1/[2\ln(a)]$. Data from one simulation with $N=500$ and $T=2000$.

equal to the corresponding degrees of freedom of the other. Extending this mechanism to spatially extended systems, one has a coupling similar to ours, but with quenched disorder (the coupling degrees of freedom). We did not perform the study of quenched disorder in diffusively coupled maps, since it implies longer spatial couplings in order to avoid the formation of walls.

Another similar system was studied by Fahy and Hammann [20]. Their subject was an ensemble of noninteracting particles in a chaotic potential. At fixed time intervals the velocities of the particles were all set equal to a Gaussian sample. If the free-fly time is small enough, all trajectories collapse into one. We performed preliminary simulations (not reported here) on a modified system, in which one reference particle followed an unperturbed trajectory, while a

replica had its velocity set equal to that of the reference at time intervals τ . Indeed, we observed a synchronization transition for small enough τ . By observing the system stroboscopically at intervals τ , we can substitute the continuous dynamics with a map. In this case the time τ controls the chaoticity of the map, while the synchronization mechanism is similar to that of Pecora and Carroll and thus to a quenched disorder. Fahy and Hammann also checked that the synchronization transition occurs if the position of the particles is set equal, instead of the velocity.

The synchronization mechanism studied in this paper is quite particular, since it implies a complete collapse of the distance between the master and the slave. The strong chaos synchronized phase is not stable with respect to the inclusion of desynchronizing effects, such as noise or nonperfect identity of parameters in the master and slave systems. On the other hand, the weak chaos transition is ruled by the exponential shrinking of difference, i.e., by negative Lyapunov exponents for the difference. Thus, it is expected that this transition is robust with respect to weak desynchronization effects, and does not depend on the threshold τ . The modification of the observed phenomena in the presence of noise will be the subject of a future work. In the present version, the synchronization transition can be considered as a mathematical tool for the definition of quantities related to the spatial propagation of chaoticity. The most natural systems to which this method can be applied are those presenting stable chaos [19], i.e., irregular behavior in the presence of negative Lyapunov spectra. For these systems an eventual small noise is wiped out by the contracting dynamics on small scales. Similar noise-free systems are those that can be approximated by cellular automata models [13].

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APPENDIX

The generation of the critical cluster can be done in a very efficient way, using a modification of the invasion percolation algorithm [21]. The random bit $r_i^t(p)$ is obtained by comparing a random number R_i^t , whose distribution probability is constant in the unit interval, with p . If $R_i^t < p$, then $r_i^t = 1$, otherwise $r_i^t = 0$.

Let us consider a lattice with the same geometry of the percolation one and assign to each site a random number R_i^t . The idea now is that of lowering p (starting from $p = 1$) until the cluster of $r_i^t = 0$ sites spans the lattice. A nice property of the directed site percolation problem (which is related to the ‘‘forcing’’ character of wetting) is that, given the random number R_i^t , the percolation cluster for p_1 is included in the percolation cluster for p_2 if $p_1 > p_2$. Thus, one has simply to choose the site with the highest R_i^t on the border of the percolation cluster (i.e., among the sites with $r_i^t = 1$ connected to some $r_j^{t-1} = 0$ site, with $|i - j| = 1$). This maximum value will become the new estimate of p , and the percolation cluster is enhanced to include all sites with $R_i^t > p$.

This procedure can be easily performed by keeping the values of the sites in the border in an ordered linked list, assuming that all sites in the $t = 0$ row are connected to a wet site.

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